Mathematical Induction

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Question

Prove, by Mathematical Induction, that

$$(n+1)^2+(n+2)^2+(n+3)^2+\ldots+(2n)^2=\frac{n(2n+1)(7n+1)}{6}$$

is true for all natural numbers n.

Discussion

Some readers may find it difficult to write the L.H.S. in P(k + 1). Some cannot factorize the L.H.S. and are forced to expand everything.

For P(1),

L.H.S. = $2^2 = 4$, R.H.S. = $\frac{1\times 3\times 8}{6}=4$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$(k+1)^2+(k+2)^2+(k+3)^2+\ldots+(2k)^2=\frac{k(2k+1)(7k+1)}{6} \quad \ldots. \quad (1)$$

For P(k + 1) ,

$$(k+2)^2+(k+3)^2+\ldots+(2k)^2+(2k+1)^2+(2k+2)^2$$ (There is a missing term in front and two more terms at the back.)

$$=(k+2)^2+(k+3)^2+\ldots+(2k)^2+(2k+1)^2+4(k+1)^2$$

$$=(k+1)^2+(k+2)^2+(k+3)^2+\ldots+(2k)^2+(2k+1)^2+3(k+1)^2$$

$$=\frac{k(2k+1)(7k+1)}{6}+(2k+1)^2+3(k+1)^2 \quad \text{, by (1)}$$

$$=\frac{(2k+1)}{6}[k(7k+1)+6(2k+1)]+3(k+1)^2$$ (Combine the first two terms)

$$=\frac{(2k+1)}{6}[7k^2+13k+6]+3(k+1)^2$$

$$=\frac{(2k+1)}{6}(7k+6)(k+1)+3(k+1)^2$$

$$=\frac{(k+1)}{6}[(2k+1)(7k+6)+18(k+1)]$$

$$=\frac{(k+1)}{6}[14k^2+37k+24]$$

$$=\frac{(k+1)}{6}(2k+3)(7k+8)=\frac{(k+1)[2(k+1)+1][7(k+1)+1]}{6}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Question

Prove, by Mathematical Induction, that

$$1 \cdot n + 2(n-1) + 3(n-2) + ... + (n-2) \cdot 2 + n \cdot 1 = \frac{1}{6} n(n+1)(n+2)$$

is true for all natural numbers n.

Discussion

The "up and down" of the L.H.S. makes it difficult to find the middle term, but you can avoid this.

Solution

Let P(n) be the proposition: $1 \cdot n + 2(n-1) + 3(n-2) + ... + (n-2) \cdot 2 + n \cdot 1 = \frac{1}{6} n(n+1)(n+2)$

For P(1),

L.H.S. = 1, R.H.S. = $\frac{1}{6} \times 1 \times 2 \times 3 = 1$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$1 \cdot k + 2(k-1) + 3(k-2) + ... + (k-2) \cdot 2 + k \cdot 1 = \frac{1}{6} k(k+1)(k+2) \qquad .... \quad (1)$$

For P(k + 1) ,

$$1 \cdot (k+1) + 2k + 3(k-1) + ... + (k-1) \cdot 3 + k \cdot 2 + (k+1) \cdot 1$$
$$= 1 \cdot (k+1) + 2[(k-1)+1] + 3[(k-2)+1] + ... + (k-1) \cdot [2+1] + k \cdot [1+1] + (k+1) \cdot 1$$
$$= 1 \cdot k + 2(k-1) + 3(k-2) + ... + (k-2) \cdot 2 + k \cdot 1$$
$$+1 \quad +2 \quad +3 \quad + ... + \quad (k-1) \quad +k \quad +(k+1)$$

(The bottom series is arithmetic)

$$= \frac{1}{6} k(k+1)(k+2) + \frac{1}{2}(k+1)(k+2) \quad \text{, by (1)}$$
$$= \frac{1}{6}(k+1)(k+2)[k+3] = \frac{1}{6}(k+1)[(k+1)+1][(k+1)+2]$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Question

Prove, by Mathematical Induction, that n(n + 1)(n + 2)(n + 3) is divisible by 24, for all natural numbers n.

Discussion

Mathematical Induction cannot be applied directly. Here we break the proposition into three parts. Also note that 24 = 4×3×2×1 = 4!

Solution

Let $P(n)$ be the proposition:

1. $n(n + 1)$ is divisible by $2! = 2$.
2. $n(n + 1)(n + 2)$ is divisible by $3! = 6$.
3. $n(n + 1)(n + 2)(n + 3)$ is divisible by $4! = 24$.

For $P(1)$,

1. $1\times2 = 2$ is divisible by 2.
2. $1\times2\times3 = 6$ is divisible by 3.
3. $1\times2\times3\times4 = 24$ is divisible by 24. $\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true for some natural number k, that is

1. $k(k + 1)$ is divisible by 2, that is, $k(k + 1) = 2a$ …. (1)
2. $k(k + 1)(k + 2)$ is divisible by 6 , that is, $k(k + 1)(k + 2) = 6b$ …. (2)
3. $k(k + 1)(k + 2)(k + 3)$ is divisible by 24 ,

   that is, $k(k + 1)(k + 2)(k + 3) = 24c$ …. (3)

   where a, b, c are natural numbers.

For $P(k + 1)$,

1. $(k + 1)(k + 2) = k(k + 1) + 2(k + 1) = 2a + 2(k + 1)$, by (1)

   $= 2 [a + k + 1]$ …. (4)

   , which is divisible by 2.
2. $(k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2)$

   $= 6b + 3 \times 2[a + k + 1]$, by (2), (4)

   $= 6 [b + a + k + 1]$ …. (5)

   , which is divisible by 6.
3. $(k + 1)(k + 2)(k + 3)(k + 4) = k(k + 1)(k + 2)(k + 3) + 4(k + 1)(k + 2)(k + 3)$

   $= 24c + 4 \times 6[b + a + k + 1]$, by (3) , (5)

   $= 24 [c + b + a + k + 1]$

   , which is divisible by 24 .

$\therefore$ $P(k + 1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers, n .

Harder Problem :

Prove, by Mathematical Induction, that $n(n + 1)(n + 2)(n + 3) \ldots(n + r - 1)$ is divisible by $r!$, for all natural numbers n, where $r = 1, 2, \ldots$.